# Mass Transfer in Laminar Flow Between Parallel Permeable Plates

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Solutions for the concentration profiles and mass transfer rates are found for various wall Sherwood numbers. Use is made of the confluent hypergeometric function in the solution, and its advantage over the previously used series solution is pointed out. Perturbation solutions are also presented for both large and small values of the wall Sherwood number.

## SCOPE

The rate of mass and heat transfer to and from fluids is of interest in a number of applications, for example, dialyzers and heat exchangers. In this work the concentration profiles and mass transfer rates are found for the laminar flow of a Newtonian fluid between parallel permeable plates where the mass flux through the plates is dependent on the permeability of the plates. The problem is analogous to the heat transfer case where the heat flux through the plates depends on the heat transmissivity of the plates.

The partial differential equation for mass transfer is solved by the method of separation of variables. Previous workers solved the resulting ordinary differential equation (6) in series form and difficulty was encountered in obtaining the higher eigenvalues of this equation, thus necessitating the use of an asymptotic solution for these eigenvalues (Sellars et al., 1956). We have used the confluent hypergeometric function in the solution of (6) and higher eigenvalues were obtained with no difficulty. Use of the confluent hypergeometric function has been made by Lauwerier (1950) for the solution of Poiseuille flow in a pipe with constant concentration boundary condition and more recently by Davis (1973) for the solution of related problems in pipes and channels. Solutions are also developed in which the eigenvalues and eigenfunctions are written as power series in $Sh_w$ (the wall Sherwood number) for small $Sh_w$ and $1/(2 + Sh_w)$ for large $Sh_w$.

## CONCLUSIONS AND SIGNIFICANCE

The solution of the mass transfer equation has been found by using the confluent hypergeometric function. Tables of eigenvalues and corresponding eigenfunctions for various wall Sherwood numbers are not presented since these can easily be found from Equations (15) and (14), respectively. Previous workers (for example, Colton et al., 1971) have calculated eigenvalues for several values of wall Sherwood number and the eigenvalues obtained from Equation (15) agreed with their results.

A perturbation solution was found for Equation (6) for the whole range of wall Sherwood number. The solutions are in good agreement with the results obtained from Equations (14) and (15) for most values of wall Sherwood number. The results were less accurate in the range $0.5 < Sh_w < 5$, but at worst provide a good initial value to the eigenvalue for use in the Newton-Raphson iteration given by Equation (16).

It is hoped that the solutions will prove useful in both heat and mass transfer problems and especially in the case of simultaneous heat and mass transfer where the simpler form of the solution to the diffusion equation may make the energy equation more amenable to a numerical solution.

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Two recent publications by Grimsrud and Babb (1966) and by Colton et al. (1971) have renewed interest in the classical Graetz problem in heat transfer and in particular in its application in mass transfer with reference to the flow of blood in dialyzers, Colton et al. (1971) give a list of references concerned with this problem in both heat and mass transfer. Grimsrud and Babb (1966) give an excellent physical description of the mass transfer process in dialyzers.

The problem considered is to find the concentration distribution and mass flux in steady laminar flow between parallel permeable plates. Separation of variables reduces the solution of the diffusion equation to an eigenvalue problem. Previous analyses have used a series solution to solve the resulting ordinary differential equation and difficulty was encountered in obtaining the higher eigenvalues which were obtained from an asymptotic solution given by Sellars et al. (1956).

In this paper the series solution is replaced by a solution involving the confluent hypergeometric function, which is itself an absolutely convergent series, from which the eigenvalues are easily calculated. We also obtain power series solutions in terms of the wall Sherwood number for the eigenvalues and corresponding eigenfunctions.

### SOLUTION

Consider a fluid with constant fluid properties flowing between parallel plates distance $2h$ apart. $z$ and $y$ are the coordinates parallel and perpendicular to the channel walls respectively. It is assumed that the flow is laminar and fully developed before the fluid contacts the section of the channel with permeable walls. Longitudinal mass diffusion is neglected in comparison with the lateral mass diffusion and steady state conditions persist throughout.

Introducing the nondimensional variables $x$, $\eta$, and $\theta$, the equation for the dimensionless concentration $\theta$ may be written as

$$\frac{3}{2}(1-\eta^2)\frac{\partial\theta}{\partial x} = \frac{\partial^2\theta}{\partial\eta^2} \tag{1}$$

with boundary conditions

$$\theta = 1 \quad x \leq 0 \quad \text{all } \eta \tag{2}$$

$$\frac{\partial\theta}{\partial\eta} = 0 \quad \eta = 0 \quad \text{all } x \tag{3}$$

$$-\frac{\partial\theta}{\partial\eta} = Sh_w\theta \quad \eta = \pm 1 \quad \text{all } x \tag{4}$$

Equation (3) is a condition for symmetry about the center line. The condition for mass balance at the wall gives (4) where the wall Sherwood number $Sh_w$ is analogous to the wall Nusselt number defined by Van der Does de Bye and Schenk (1952). $Sh_w = 0$ is the constant flux boundary condition and $Sh_w = \infty$ the constant concentration boundary condition.

The solution of (1) obtained by separation of variables is

$$\theta = \sum_j E_j M_j(\eta) e^{-2/3\,\gamma_j^2 x} \tag{5}$$

where $M_j(\eta)$ and $\gamma_j$ are the eigenfunctions and corresponding eigenvalues of

$$M_j''(\eta) + \gamma_j^2(1-\eta^2)M_j(\eta) = 0 \tag{6}$$

subject to

$$M_j(0) = 1 \quad M_j'(0) = 0 \tag{7}$$

$$-M_j'(1) = Sh_w M_j(1) \tag{8}$$

where $'$ denotes differentiation with respect to $\eta$.

The eigenconstants $E_j$ are obtained by applying the initial condition (2) on $\theta$ and using the usual orthogonality relationships.

Thus

$$E_j = \int_{-1}^{1}(1-\eta^2)M_j(\eta)d\eta \Bigg/ \int_{-1}^{1}(1-\eta^2)M_j^2(\eta)d\eta \tag{9}$$

$$= -2 \Bigg/ \gamma_j\left\{\frac{\partial M_j}{\partial\gamma_j} + \frac{1}{Sh_w}\frac{\partial M_j'}{\partial\gamma_j}\right\}_{\eta=1} \quad \gamma_j \neq 0 \tag{10}$$

$$E_0 = 1 \quad E_j = 0 \quad j = 1, 2, \ldots \quad \gamma_j = 0$$

In (6) write $x = \gamma_j\eta^2$ and $M_j(x) = v(x)/x^{1/4}$, then the equation becomes

$$v''(x) + \left\{-\frac{1}{4} + \frac{k}{x} + \frac{\left(\frac{1}{4} - m^2\right)}{x^2}\right\}v(x) = 0 \tag{11}$$

where $k = \gamma_j/4$ and $m = \pm 1/4$. This is Whittaker's equation, the solution of which is (from Whittaker and Watson, p. 337)

$$M_{k,m}(x) = x^{1/2+m}e^{-1/2x}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}+m-k\right)_n x^n}{(2m+1)_n\, n!} \tag{12}$$

where $(a)_n = a(a+1)\ldots(a+n-1)$.

$M_{k,m}(x)$ is the confluent hypergeometric function. The summation is absolutely convergent for all $x$ provided $(2m+1) \neq 0, -1, -2, \ldots$.

In order to satisfy the initial conditions (7) we find that m $= -1/4$. Thus

$$M_j(\eta) = \exp\left(-\frac{\gamma_j\eta^2}{2}\right)\sum_{n=0}^{\infty}\frac{\left(\frac{1}{4}-\frac{\gamma_j}{4}\right)_n}{\left(\frac{1}{2}\right)_n n!}\gamma_j^n\eta^{2n} \tag{13}$$

$$= \exp\left(-\frac{\gamma_j\eta^2}{2}\right)\left\{1 + \sum_{n=1}^{\infty}\frac{(1-\gamma_j)\ldots(4n-3-\gamma_j)}{(2n)!}\gamma_j^n\eta^{2n}\right\} \tag{14}$$

The eigenvalues of (14) are found by application of (8) and are the roots of

$$F(\gamma) = (Sh_w - \gamma) + \sum_{n=1}^{\infty}\frac{(1-\gamma)\ldots(4n-3-\gamma)}{(2n)!}\gamma^n(Sh_w + 2n - \gamma) = 0 \tag{15}$$

where the suffix $j$ has been omitted. Equation (15) can be solved by using a Newton-Raphson technique, the required iterative scheme being

$$\gamma^{(i+1)} = \gamma^{(i)} - F(\gamma^{(i)})/\partial F/\partial\gamma \tag{16}$$

where

$$\frac{\partial F}{\partial\gamma} = -1 + \sum_{n=1}^{\infty}\frac{(1-\gamma)\ldots(4n-3-\gamma)}{(2n)!}\gamma^n\left[\left(\frac{n}{\gamma} - \sum_{p=1}^{n}\frac{1}{(4p-3-\gamma)}\right)\times(Sh_w + 2n - \gamma) - 1\right] \tag{17}$$

The initial guess $\gamma^{(0)}$ in (16) can be found by using (15), for example, take $Sh_w = 0$ then (15) is

$$F(\gamma) = -\gamma + \frac{\gamma(1-\gamma)(2-\gamma)}{2} + \frac{\gamma^2(1-\gamma)(5-\gamma)(4-\gamma)}{24} + \ldots = 0$$

and $\gamma = 0$ is a root. Now write

$$F_1(\gamma) = -1 + \frac{(1-\gamma)(2-\gamma)}{2} + \frac{\gamma(1-\gamma)(5-\gamma)(4-\gamma)}{24} + \ldots$$

then $F_1(1) = -1$, $F_1(5) = 5$, $F_1(9) = -33$ and so on. The remaining eigenvalues can be bracketed immediately, and in this case we have (1, 5), (5, 9), (9, 13) etc. as the intervals in which roots of (15) occur for $Sh_w = 0$. Similar intervals are obtained for each value of $Sh_w$.

This analysis suggests that the difference between successive eigenvalues is approximately 4. In practice, the numerical scheme adds 4 to the current eigenvalue and iterates rapidly to the next eigenvalue. Clearly any good initial guess at an eigenvalue will provide a convergent iterative scheme.

The above method has been used successfully to obtain the first 20 eigenvalues for a wide range of $Sh_w$ and in each case convergence has been rapid. In order to retain six decimal place accuracy the number of terms of (15) which are used increases with increasing number of eigenvalue; for example, 100 terms of (15) were necessary to obtain the twentieth eigenvalue. The number of terms required in (14) to give $M_j(\eta)$ correct to six decimal places is much less than the number of terms involved in the calculation of $\gamma_j$.

To complete the solution for the dimensionless concentration $\theta$, the eigenconstants $E_j$ are required. These are found from (10) after the required derivatives of $M_j(\eta)$ have been calculated from (14).

It is useful at this stage to define the mixing cup concentration $\theta_m$ by

$$\theta_m = \int_{-1}^{1} \frac{3}{2}(1-\eta^2)\theta d\eta \Bigg/ \int_{-1}^{1} \frac{3}{2}(1-\eta^2) d\eta$$

$$= \sum_j B_j \exp\left(-\frac{2}{3}\gamma_j^2 x\right) \quad (18)$$

where

$$B_j = -3E_j M_j'(1)/2\gamma_j^2 \quad (19)$$

The overall Sherwood number is defined by

$$Sh_o = -\frac{1}{\theta_m}\left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=1} = -\frac{1}{\theta_m}\left(\frac{d\theta_m}{dx}\right)$$

$$= \frac{\frac{2}{3}\sum_j B_j\gamma_j^2 e^{-2/3\,\gamma_j^2 x}}{\sum_j B_j e^{-2/3\,\gamma_j^2 x}} \quad (20)$$

As $x \to \infty$, only the first term in each of the summations in (20) will remain and thus the limiting overall Sherwood number is given as

$$Sh_{o\infty} = \frac{2}{3}\gamma_0^2 \quad (21)$$

A local fluid-side Sherwood number may be defined by

TABLE 1. COEFFICIENTS IN POWER SERIES FOR $\epsilon = Sh_w$

$\gamma_0^2 = 1.5\epsilon - 0.728571\epsilon^2 + 0.323340\epsilon^3 - 0.129664\epsilon^4 + 0.045748\epsilon^5$
$E_0 = 1.0 + 0.139286\epsilon - 0.074665\epsilon^2 + 0.032105\epsilon^3 - 0.010375\epsilon^4$

| $\eta$ | $M_{00}$ | $M_{01}$ | $M_{02}$ | $M_{03}$ | $M_{04}$ | $M_{05}$ |
|---|---|---|---|---|---|---|
| 0 | 1.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.1 | 1.00000 | −0.007487 | 0.003646 | −0.001623 | 0.000653 | −0.000232 |
| 0.2 | 1.00000 | −0.029800 | 0.014621 | −0.006567 | 0.002675 | −0.000966 |
| 0.3 | 1.000000 | −0.066487 | 0.033022 | −0.015042 | 0.006237 | −0.002310 |
| 0.4 | 1.000000 | −0.116800 | 0.058954 | −0.027354 | 0.011604 | −0.004435 |
| 0.5 | 1.000000 | −0.179687 | 0.092466 | −0.043834 | 0.019081 | −0.007545 |
| 0.6 | 1.000000 | −0.253800 | 0.133439 | −0.064745 | 0.028956 | −0.011848 |
| 0.7 | 1.000000 | −0.337487 | 0.181478 | −0.090163 | 0.041413 | −0.017500 |
| 0.8 | 1.000000 | −0.428800 | 0.235767 | −0.119832 | 0.056424 | −0.024544 |
| 0.9 | 1.000000 | −0.525487 | 0.294937 | −0.153008 | 0.073627 | −0.032822 |
| 1.0 | 1.000000 | −0.625000 | 0.356920 | −0.188300 | 0.092194 | −0.041887 |

$\gamma_1^2 = 18.380297 + 4.084622\epsilon - 0.886126\epsilon^2 + 0.076819\epsilon^3 + 0.047037\epsilon^4 - 0.032721\epsilon^5$

| $\eta$ | $M_{10}$ | $M_{11}$ | $M_{12}$ | $M_{13}$ | $M_{14}$ | $M_{15}$ |
|---|---|---|---|---|---|---|
| 0 | 1.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.1 | 0.909[illegible]44 | −0.019772 | 0.004357 | −0.000401 | −0.000222 | 0.000159 |
| 0.2 | 0.656420 | −0.071674 | 0.016564 | −0.001794 | −0.000736 | 0.000588 |
| 0.3 | 0.288871 | −0.136243 | 0.034134 | −0.004606 | −0.001144 | 0.001148 |
| 0.4 | −0.126969 | −0.189075 | 0.053339 | −0.009179 | −0.000950 | 0.001640 |
| 0.5 | −0.524501 | −0.209222 | 0.069880 | −0.015442 | 0.000254 | 0.001852 |
| 0.6 | −0.851829 | −0.185495 | 0.079826 | −0.022769 | 0.002640 | 0.001611 |
| 0.7 | −1.081210 | −0.118495 | 0.080574 | −0.030136 | 0.006091 | 0.000830 |
| 0.8 | −1.211115 | −0.018213 | 0.071541 | −0.036507 | 0.010263 | −0.000467 |
| 0.9 | −1.262117 | 0.101245 | 0.054336 | −0.041312 | 0.014742 | −0.002154 |
| 1.0 | −1.269692 | 0.227513 | 0.032428 | −0.044834 | 0.019242 | −0.004045 |

$\gamma_2^2 = 68.951816 + 5.001182\epsilon - 0.676435\epsilon^2 + 0.015820\epsilon^3 + 0.021464\epsilon^4 - 0.006549\epsilon^5$

| $\eta$ | $M_{20}$ | $M_{21}$ | $M_{22}$ | $M_{23}$ | $M_{24}$ | $M_{25}$ |
|---|---|---|---|---|---|---|
| 0 | 1.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.1 | 0.675087 | −0.022201 | 0.003100 | −0.000097 | −0.000093 | 0.000030 |
| 0.2 | −0.085685 | −0.059993 | 0.009346 | −0.000532 | −0.000224 | 0.000087 |
| 0.3 | −0.796422 | −0.056958 | 0.011857 | −0.001379 | −0.000111 | 0.000102 |
| 0.4 | −1.037215 | 0.012460 | 0.005158 | −0.002086 | 0.000338 | 0.000016 |
| 0.5 | −0.706257 | 0.115302 | −0.010380 | −0.001629 | 0.000877 | −0.000158 |
| 0.6 | −0.020146 | 0.186872 | −0.028280 | 0.000627 | 0.001085 | −0.000342 |
| 0.7 | 0.684396 | 0.180941 | −0.040076 | 0.004246 | 0.000686 | −0.000434 |
| 0.8 | 1.165628 | 0.096957 | −0.040562 | 0.007977 | −0.000268 | −0.000377 |
| 0.9 | 1.370908 | −0.032108 | −0.030266 | 0.010720 | −0.001498 | −0.000185 |
| 1.0 | 1.402190 | −0.172356 | −0.014005 | 0.012374 | −0.002750 | 0.000079 |

(continued on next page)

TABLE 1 (continued)

$\gamma_3^2 = 151.550771 + 5.663203\epsilon - 0.583789\epsilon^2 + 0.006213\epsilon^3 + 0.011651\epsilon^4 - 0.002419\epsilon^5$

| $\eta$ | $M_{30}$ | $M_{31}$ | $M_{32}$ | $M_{33}$ | $M_{34}$ | $M_{35}$ |
|---|---|---|---|---|---|---|
| 0 | 1.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.1 | 0.334091 | −0.021669 | 0.002348 | −0.000047 | −0.000043 | 0.000010 |
| 0.2 | −0.777899 | −0.029398 | 0.004114 | −0.000266 | −0.000043 | 0.000016 |
| 0.3 | −0.897091 | 0.033405 | −0.001690 | −0.000383 | 0.000110 | −0.000010 |
| 0.4 | 0.090887 | 0.092862 | −0.010797 | 0.000254 | 0.000210 | −0.000049 |
| 0.5 | 0.996042 | 0.043792 | −0.010921 | 0.001400 | 0.000004 | −0.000045 |
| 0.6 | 0.884759 | −0.087178 | 0.002440 | 0.001617 | −0.000375 | 0.000024 |
| 0.7 | −0.032881 | −0.171615 | 0.019598 | −0.000005 | −0.000524 | 0.000108 |
| 0.8 | −0.961187 | −0.133369 | 0.027408 | −0.002634 | −0.000238 | 0.000137 |
| 0.9 | −1.418823 | −0.005571 | 0.021873 | −0.004749 | 0.000325 | 0.000093 |
| 1.0 | −1.491570 | 0.144303 | 0.008618 | −0.005863 | 0.000924 | 0.000007 |

$\gamma_4^2 = 266.162305 + 6.196733\epsilon - 0.527307\epsilon^2 + 0.003162\epsilon^3 + 0.007393\epsilon^4 - 0.001195\epsilon^5$

| $\eta$ | $M_{40}$ | $M_{41}$ | $M_{42}$ | $M_{43}$ | $M_{44}$ | $M_{45}$ |
|---|---|---|---|---|---|---|
| 0 | 1.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.1 | −0.059587 | −0.018956 | 0.001734 | −0.000031 | −0.000022 | 0.000004 |
| 0.2 | −1.004850 | 0.003899 | 0.000359 | −0.000125 | 0.000013 | 0.000001 |
| 0.3 | 0.115106 | 0.057011 | −0.005363 | 0.000092 | 0.000070 | −0.000013 |
| 0.4 | 1.041350 | −0.005517 | −0.002335 | 0.000512 | −0.000039 | −0.000005 |
| 0.5 | 0.042496 | −0.097306 | 0.008668 | 0.000013 | −0.000148 | 0.000023 |
| 0.6 | −1.082731 | −0.028735 | 0.008753 | −0.001107 | 0.000017 | 0.000022 |
| 0.7 | −0.633033 | 0.119963 | −0.006270 | −0.000947 | 0.000262 | −0.000022 |
| 0.8 | 0.641310 | 0.146912 | −0.019015 | 0.000823 | 0.000228 | −0.000057 |
| 0.9 | 1.427429 | 0.030500 | −0.017362 | 0.002590 | −0.000077 | −0.000047 |
| 1.0 | 1.560068 | −0.126606 | −0.006112 | 0.003454 | −0.000433 | −0.000008 |

$\gamma_5^2 = 412.779539 + 6.650353\epsilon - 0.487824\epsilon^2 + 0.001853\epsilon^3 + 0.005160\epsilon^4 - 0.000694\epsilon^5$

| $\eta$ | $M_{50}$ | $M_{51}$ | $M_{52}$ | $M_{53}$ | $M_{54}$ | $M_{55}$ |
|---|---|---|---|---|---|---|
| 0 | 1.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.1 | −0.443952 | −0.014678 | 0.001195 | −0.000022 | −0.000011 | 0.000002 |
| 0.2 | −0.630186 | 0.025655 | −0.001652 | −0.000033 | 0.000023 | −0.000002 |
| 0.3 | 0.984795 | 0.013510 | −0.002195 | 0.000185 | 0.000003 | −0.000003 |
| 0.4 | −0.059276 | −0.066336 | 0.005252 | −0.000034 | −0.000059 | 0.000008 |
| 0.5 | −1.025268 | 0.024969 | 0.001205 | −0.000488 | 0.000047 | 0.000001 |
| 0.6 | 0.466244 | 0.093180 | −0.009174 | 0.000257 | 0.000086 | −0.000016 |
| 0.7 | 1.082025 | −0.048892 | −0.002095 | 0.000954 | −0.000097 | −0.000002 |
| 0.8 | −0.247950 | −0.144262 | 0.012782 | −0.000063 | −0.000170 | 0.000025 |
| 0.9 | −1.405011 | −0.048481 | 0.014428 | −0.001570 | 0.000004 | 0.000026 |
| 1.0 | −1.616080 | 0.114146 | 0.004684 | −0.002292 | 0.000242 | 0.000006 |

$$Sh_f = -\frac{1}{(\theta_m - \theta_w)}\left(\frac{\partial \theta}{\partial \eta}\right)_{\eta=1} \quad (22)$$

and is related to the overall Sherwood number by the expression

$$\frac{1}{Sh_o} = \frac{1}{Sh_f} + \frac{1}{Sh_w} \quad (23)$$

## POWER SERIES SOLUTION

For each value of wall Sherwood number, Equation (15) has to be solved and Equation (14) evaluated to produce the required eigenvalues and eigenfunctions. A solution will now be obtained where evaluation of an eigenvalue takes place once only for each case of $Sh_w \leq 1$ and $Sh_w \geq 1$.

For small $Sh_w$, the form of the wall boundary condition (8) suggests a solution of the form

$$M_j(\eta) = M_{j0}(\eta) + \epsilon M_{j1}(\eta) + \epsilon^2 M_{j2}(\eta) + \ldots \quad (24)$$

and

$$\gamma_j^2 = \lambda_j^2 + \epsilon\alpha_{j1} + \epsilon^2\alpha_{j2} + \ldots \quad (25)$$

where $\epsilon = Sh_w$.

In fact, examination of the results obtained for various $Sh_w$ suggested that the forms (24) and (25) were also applicable for large $Sh_w$ with $\epsilon = 1/Sh_w$.

Substituting for $M_j(\eta)$ and $\gamma_j^2$ from (24) and (25) into Equation (6) and equating like powers of $\epsilon$ gives the following system of equations:

$$M''_{j0}(\eta) + \lambda_j^2(1-\eta^2)M_{j0}(\eta) = 0 \quad (26)$$

$$M''_{j1}(\eta) + \lambda_j^2(1-\eta^2)M_{j1}(\eta) = -\alpha_{j1}(1-\eta^2)M_{j0}(\eta) \quad (27)$$

$$M''_{jn}(\eta) + \lambda_j^2(1-\eta^2)M_{jn}(\eta) = -\alpha_{j1}(1-\eta^2)M_{jn-1}(\eta) - \ldots - \alpha_{jn}(1-\eta^2)M_{j0}(\eta) \quad n = 2, 3, \ldots \quad (28)$$

The boundary conditions become

At $\eta = 0$

$$M_{j0}(0) = 1 \quad M_{ji}(0) = 0 \quad i = 1, 2, 3, \ldots \quad (29)$$

$$M'_{ji}(0) = 0 \quad i = 0, 1, 2, \ldots \quad (30)$$

At $\eta = 1$

$$M'_{j0}(1) = 0 \quad M'_{ji}(1) = -M_{ji-1}(1)$$
$$i = 1, 2, 3, \ldots \quad \text{for} \quad Sh_w < 1 \quad (31)$$
$$M_{j0}(1) = 0 \quad M_{ji}(1) = -M'_{ji-1}(1)$$
$$i = 1, 2, 3, \ldots \quad \text{for} \quad Sh_w > 1 \quad (32)$$

The system of Equations (26) to (28), subject to the boundary conditions, was solved numerically using a Milne-Simpson technique. For small $Sh_w$ the results are given in Table 1 and were accurate for $Sh_w < ½$ and provided useful results up to $Sh_w = 1$.

For large $Sh_w$ the terms in the series obtained appeared to be asymptotic. Nevertheless, the results were in good agreement for $Sh_w \geqq 10$. However, in an attempt to stretch the region for which the series is useful, that is, for $Sh_w < 10$, we supposed that $\epsilon = 1/(k + Sh_w)$ where

TABLE 2. DERIVATIVES OF COEFFICIENTS AT $\eta = 1$ FOR $\epsilon = Sh_w$

| $n$ | $M'_{0n}(1)$ | $E_{0n}$ | $M'_{1n}(1)$ | $(\partial M_{1n}/\partial \gamma_1)_{\eta=1}$ | $(\partial M'_{1n}/\partial \gamma_1)_{\eta=1}$ |
|---|---|---|---|---|---|
| 0 | 0.0 | 1.0 | 0.0 | 0.477596 | 2.665341 |
| 1 | −1.000000 | 0.139286 | 1.269692 | 0.396436 | 0.497413 |
| 2 | 0.625000 | −0.074665 | −0.227513 | −0.136618 | −0.325618 |
| 3 | −0.356920 | 0.032105 | −0.032428 | 0.025037 | 0.105111 |
| 4 | 0.188300 | 0.010375 | 0.044834 | 0.004331 | −0.010761 |
| 5 | −0.092194 | — | −0.019242 | −0.005988 | −0.009784 |

| $n$ | $M'_{2n}(1)$ | $(\partial M_{2n}/\partial \gamma_2)_{\eta=1}$ | $(\partial M'_{2n}/\partial \gamma_2)_{\eta=1}$ | $M'_{3n}(1)$ | $(\partial M_{3n}/\partial \gamma_3)_{\eta=1}$ | $(\partial M'_{3n}/\partial \gamma_3)_{\eta=1}$ |
|---|---|---|---|---|---|---|
| 0 | 0.0 | −0.572342 | −4.656260 | 0.0 | 0.627368 | 6.484711 |
| 1 | −1.402190 | −0.268596 | −0.283745 | 1.491570 | 0.216002 | 0.203373 |
| 2 | 0.172356 | 0.055854 | 0.174368 | −0.144303 | −0.033921 | −0.128882 |
| 3 | 0.014005 | −0.004235 | −0.037347 | −0.008618 | 0.001653 | 0.022281 |
| 4 | −0.012374 | −0.001615 | 0.001311 | 0.005863 | 0.000654 | −0.000601 |
| 5 | 0.002750 | 0.000722 | 0.001841 | −0.000924 | −0.000194 | −0.000645 |

| $n$ | $M'_{4n}(1)$ | $(\partial M_{4n}/\partial \gamma_4)_{\eta=1}$ | $(\partial M'_{4n}/\partial \gamma_4)_{\eta=1}$ | $M'_{5n}(1)$ | $(\partial M_{5n}/\partial \gamma_5)_{\eta=1}$ | $(\partial M'_{5n}/\partial \gamma_5)_{\eta=1}$ |
|---|---|---|---|---|---|---|
| 0 | 0.0 | −0.666644 | −8.214556 | 0.0 | 0.697435 | 9.874319 |
| 1 | −1.560068 | −0.185583 | −0.160361 | 1.616080 | 0.165170 | 0.133296 |
| 2 | 0.126606 | 0.023957 | 0.106050 | −0.114146 | −0.018335 | −0.091942 |
| 3 | 0.006112 | −0.000879 | −0.015696 | −0.004684 | 0.000547 | 0.012019 |
| 4 | −0.003454 | −0.000332 | 0.000368 | 0.002292 | 0.000194 | −0.000254 |
| 5 | 0.000433 | 0.000076 | 0.000308 | −0.000242 | −0.000037 | −0.000174 |

TABLE 3. COEFFICIENTS IN POWER SERIES FOR $\epsilon = 1/(2 + Sh_w)$

$\gamma_0^2 = 2.827763 - 4.852930\epsilon - 2.250348\epsilon^2 + 0.553284\epsilon^3 + 1.553089\epsilon^4 + 0.471264\epsilon^5$

| $\eta$ | $M_{00}$ | $M_{01}$ | $M_{02}$ | $M_{03}$ | $M_{04}$ | $M_{05}$ |
|---|---|---|---|---|---|---|
| 0 | 1.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.1 | 0.985918 | 0.024111 | 0.011278 | −0.002658 | −0.007717 | −0.002414 |
| 0.2 | 0.944343 | 0.094626 | 0.045403 | −0.009365 | −0.030289 | −0.010322 |
| 0.3 | 0.877224 | 0.206339 | 0.103111 | −0.016528 | −0.065983 | −0.025545 |
| 0.4 | 0.787600 | 0.351341 | 0.185205 | −0.018835 | −0.111952 | −0.050557 |
| 0.5 | 0.679303 | 0.520071 | 0.292026 | −0.010194 | −0.164381 | −0.087782 |
| 0.6 | 0.556603 | 0.702574 | 0.422929 | 0.015115 | −0.218832 | −0.138851 |
| 0.7 | 0.423798 | 0.889835 | 0.575882 | 0.061198 | −0.270854 | −0.203999 |
| 0.8 | 0.284819 | 1.075027 | 0.747292 | 0.129323 | −0.316881 | −0.281758 |
| 0.9 | 0.142850 | 1.254600 | 0.932082 | 0.216974 | −0.355377 | −0.369012 |
| 1.0 | 0.000000 | 1.429155 | 1.123972 | 0.316797 | −0.388094 | −0.461381 |

$\gamma_1^2 = 32.147278 - 36.613353\epsilon + 0.703297\epsilon^2 + 68.523658\epsilon^3 - 16.151265\epsilon^4 - 188.827428\epsilon^5$

| $\eta$ | $M_{10}$ | $M_{11}$ | $M_{12}$ | $M_{13}$ | $M_{14}$ | $M_{15}$ |
|---|---|---|---|---|---|---|
| 0 | 1.0 | 0.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.1 | 0.843772 | 0.173154 | 0.002057 | −0.295832 | 0.057996 | 0.897779 |
| 0.2 | 0.426158 | 0.582856 | 0.065869 | −0.994431 | −0.006181 | 3.059269 |
| 0.3 | −0.120471 | 0.961619 | 0.304870 | −1.616888 | −0.679921 | 5.072869 |
| 0.4 | −0.634506 | 1.030943 | 0.761350 | −1.619867 | −2.204255 | 5.183040 |
| 0.5 | −0.983217 | 0.643927 | 1.321336 | −0.656774 | −4.214084 | 2.056550 |
| 0.6 | −1.101348 | −0.155467 | 1.744573 | 1.253765 | −5.809371 | −4.537379 |
| 0.7 | −0.997324 | −1.178147 | 1.789012 | 3.742603 | −6.016438 | −13.538437 |
| 0.8 | −0.731087 | −2.200927 | 1.339671 | 6.284894 | −4.314536 | −23.103445 |
| 0.9 | −0.378731 | −3.077930 | 0.458511 | 8.477690 | −0.885138 | −31.615748 |
| 1.0 | −0.000000 | −3.807069 | −0.661124 | 10.275055 | 3.524045 | −38.721624 |

(continued on next page)

Table 3 (continued)

$\gamma_2^2 = 93.474852 - 89.000159\epsilon + 47.719870\epsilon^2 + 310.599894\epsilon^3 - 572.684884\epsilon^4$

| $\eta$ | $M_{20}$ | $M_{21}$ | $M_{22}$ | $M_{23}$ | $M_{24}$ |
|---|---|---|---|---|---|
| 0 | 1.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.1 | 0.568534 | 0.378401 | −0.173001 | −1.351715 | 2.233404 |
| 0.2 | −0.351258 | 0.863676 | −0.112576 | −3.343734 | 3.140438 |
| 0.3 | −0.984308 | 0.379577 | 0.791431 | −2.036974 | −4.729105 |
| 0.4 | −0.841406 | −1.100015 | 1.671973 | 3.775044 | −15.723178 |
| 0.5 | −0.074979 | −2.345427 | 0.888179 | 10.290430 | −14.210594 |
| 0.6 | 0.753970 | −2.108813 | −1.860588 | 11.493535 | 7.161513 |
| 0.7 | 1.166875 | −0.279250 | −4.894320 | 4.421973 | 37.556181 |
| 0.8 | 1.049900 | 2.211070 | −6.145417 | −8.519427 | 57.992553 |
| 0.9 | 0.583115 | 4.364027 | −4.948701 | −22.329313 | 58.371521 |
| 1.0 | −0.000000 | 5.920229 | −2.230060 | −34.213739 | 43.923428 |

$\gamma_3^2 = 186.804503 - 158.399499\epsilon + 161.138196\epsilon^2 + 822.582808\epsilon^3 - 2949.534884\epsilon^4$

| $\eta$ | $M_{30}$ | $M_{31}$ | $M_{32}$ | $M_{33}$ | $M_{34}$ |
|---|---|---|---|---|---|
| 0 | 1.0 | 0.0 | 0.0 | 0.0 | 0.0 |
| 0.1 | 0.203563 | 0.567115 | −0.490999 | −3.115109 | 9.742230 |
| 0.2 | −0.921271 | 0.476343 | 0.224910 | −3.722158 | 1.664970 |
| 0.3 | −0.634255 | −1.373098 | 2.033778 | 6.772837 | −34.046894 |
| 0.4 | 0.601588 | −1.920393 | 0.024717 | 14.704535 | −18.944536 |
| 0.5 | 1.035007 | 0.782739 | −4.592928 | 1.059399 | 59.065163 |
| 0.6 | 0.174331 | 3.575958 | −3.810023 | −24.581923 | 84.413045 |
| 0.7 | −0.914199 | 2.728854 | 3.994477 | −31.286451 | −11.727784 |
| 0.8 | −1.232889 | −1.222843 | 11.174590 | −6.364573 | −148.253161 |
| 0.9 | −0.765618 | −5.237307 | 11.639907 | 33.536443 | −215 224629 |
| 1.0 | 0.000000 | −7.892490 | 6.911067 | 70.135470 | −203.277949 |

$\gamma_4^2 = 312.133917 - 242.903893\epsilon + 356.351720\epsilon^2$

| $\eta$ | $M_{40}$ | $M_{41}$ | $M_{42}$ |
|---|---|---|---|
| 0 | 1.0 | 0.0 | 0.0 |
| 0.1 | −0.193678 | 0.674527 | −0.813017 |
| 0.2 | −0.941396 | −0.499895 | 1.512793 |
| 0.3 | 0.501499 | −1.810906 | 1.272758 |
| 0.4 | 0.863168 | 1.569399 | −5.079520 |
| 0.5 | −0.612123 | 2.897321 | −0.388338 |
| 0.6 | −0.971294 | −2.118422 | 9.922012 |
| 0.7 | 0.340926 | −4.936812 | 3.035100 |
| 0.8 | 1.270792 | −0.588251 | −14.040992 |
| 0.9 | 0.928480 | 5.738225 | −19.874618 |
| 1.0 | −0.000000 | 9.770776 | −13.065435 |

$\gamma_5^2 = 469.460545 - 341.252099\epsilon + 645.452225\epsilon^2$

| $\eta$ | $M_{50}$ | $M_{51}$ | $M_{52}$ |
|---|---|---|---|
| 0 | 1.0 | 0.0 | 0.0 |
| 0.1 | −0.560648 | 0.653065 | −0.943601 |
| 0.2 | −0.398569 | −1.451442 | 2.969185 |
| 0.3 | 1.016105 | 0.289694 | −3.242439 |
| 0.4 | −0.573759 | 2.671638 | −1.879255 |
| 0.5 | −0.627136 | −3.281080 | 10.054238 |
| 0.6 | 1.036021 | −1.842172 | −6.967079 |
| 0.7 | 0.350495 | 5.649846 | −14.080497 |
| 0.8 | −1.165115 | 2.942521 | 12.777021 |
| 0.9 | −1.071943 | −5.885431 | 29.055380 |
| 1.0 | 0.000000 | −11.579366 | 20.497567 |

$k$ is a constant. Application of the wall boundary condition gives

$$\epsilon(kM_j(1) - M'_j(1)) = M_j(1) \qquad (33)$$

Thus, provided $\epsilon k = 0(\epsilon)$ we can write the boundary condition as

$$M_{j0}(1) = 0$$
$$M_{j1}(1) = kM_{j0}(1) - M'_{j0}(1) = -M'_{j0}(1) \qquad (34)$$
$$M_{jn}(1) = kM_{jn-1}(1) - M'_{jn-1}(1) \quad n = 1, 2, 3, \ldots$$

Numerical solutions were obtained for $k = 0$, 1, 2, and 3 and it was found that for $k = 2$ the range of applicability of the power series was greatest. Indeed, for $Sh_w = 5$ the series gave results in close agreement with those obtained from Equations (14) and (15). For $Sh_w$ as low as unity the series gave good approximations to the eigenvalues and eigenfunctions. The results for the first six eigenvalues and eigenfunctions corresponding to the case $k = 2$ are given in Table 3.

In evaluating the eigenconstants $E_j$, the terms $(\partial M_j/$

TABLE 4. DERIVATIVES OF COEFFICIENTS AT $\eta = 1$ FOR $\epsilon = 1/(2 + Sh_w)$

| $n$ | $M'_{0n}(1)$ | $(\partial M_{0n}/\partial\gamma_0)_{\eta=1}$ | $(\partial M'_{0n}/\partial\gamma_0)_{\eta=1}$ | $M'_{1n}(1)$ | $(\partial M_{1n}/\partial\gamma_1)_{\eta=1}$ | $(\partial M'_{1n}/\partial\gamma_1)_{\eta=1}$ |
|---|---|---|---|---|---|---|
| 0 | −1.429155 | −0.990437 | −1.201936 | 3.807069 | 1.179107 | 2.153455 |
| 1 | 1.734339 | 0.210554 | −0.530593 | −6.953014 | −0.216639 | 6.040141 |
| 2 | 1.931146 | 0.902850 | 1.174378 | −11.597303 | −3.926248 | −8.971932 |
| 3 | 1.021689 | 1.355407 | 2.609119 | 17.026065 | −1.287190 | −20.071813 |
| 4 | −0.314808 | 1.386347 | 2.926116 | 45.769714 | 14.354840 | 33.207037 |
| 5 | −0.932526 | 1.311467 | 2.482050 | −67.519718 | 5.659865 | 77.868032 |

| $n$ | $M'_{2n}(1)$ | $(\partial M_{2n}/\partial\gamma_2)_{\eta=1}$ | $(\partial M'_{2n}/\partial\gamma_2)_{\eta=1}$ | $M'_{3n}(1)$ | $(\partial M_{3n}/\partial\gamma_3)_{\eta=1}$ | $(\partial M'_{3n}/\partial\gamma_3)_{\eta=1}$ |
|---|---|---|---|---|---|---|
| 0 | −5.920229 | −1.286249 | −3.057009 | 7.892490 | 1.362021 | 3.916698 |
| 1 | 14.070518 | 0.202043 | −14.751610 | −22.696046 | −0.191628 | 25.744455 |
| 2 | 29.753619 | 8.406395 | 31.125503 | −56.313337 | −14.004469 | −71.365255 |
| 3 | −112.350907 | −6.638292 | 65.653820 | 343.548889 | 25.169551 | −141.777142 |
| 4 | −172.624066 | −68.388291 | −323.080347 | 285.625657 | 165.883938 | 1250.402992 |

| $n$ | $M'_{4n}(1)$ | $(\partial M_{4n}/\partial\gamma_4)_{\eta=1}$ | $(\partial M'_{4n}/\partial\gamma_4)_{\eta=1}$ | $M'_{5n}(1)$ | $(\partial M_{5n}/\partial\gamma_5)_{\eta=1}$ | $(\partial M'_{5n}/\partial\gamma_5)_{\eta=1}$ |
|---|---|---|---|---|---|---|
| 0 | −9.770776 | −1.421330 | −4.743257 | 11.579366 | 1.470412 | 5.543718 |
| 1 | 32.606987 | 0.183914 | −38.615595 | −43.656297 | −0.177869 | 53.122282 |
| 2 | 91.236978 | 20.529685 | 131.824582 | −134.527612 | −27.864451 | −214.027123 |

$\partial\gamma_j)_{\eta=1}$ and $(\partial M'_j/\partial\gamma_j)_{\eta=1}$ are required.

Writing $Y(\eta) = \partial M_j(\eta)/\partial\gamma_j$ and differentiating (6) with respect to $\gamma_j$ gives

$$Y_j''(\eta) + \gamma_j^2(1-\eta^2)Y_j(\eta) = -2\gamma_j(1-\eta^2)M_j(\eta) \quad (35)$$

As before, write

$$Y_j(\eta) = Y_{j0}(\eta) + \epsilon Y_{j1}(\eta) + \epsilon^2 Y_{j2}(\eta) + \ldots \quad (36)$$

and

$$\gamma_j = \lambda_j + \epsilon\beta_{j1} + \epsilon^2\beta_{j2} + \ldots \quad (37)$$

where, for example, $\beta_{j1} = \alpha_{j1}/2\lambda_j$ is obtained from the expression (25) for $\gamma_j^2$.

Substituting for $Y_j(\eta)$, $\gamma_j^2$ and $\gamma_j$ in (35) and equating the coefficients of like powers of $\epsilon$ gives a system of differential equations similar to the system (26) to (28) and subject to the initial conditions

$$Y_{ji}(0) = 0 \quad Y'_{ji}(0) = 0 \quad i = 0, 1, 2, \ldots \quad (38)$$

This system of equations is solved simultaneously with the system (26) to (28). The required values of $(\partial M_j/\partial\gamma_j)_{\eta=1}$ and $(\partial M'_j/\partial\gamma_j)_{\eta=1}$ are given in Table 2 for $\epsilon = Sh_w$ and in Table 4 for $\epsilon = 1/(2 + Sh_w)$. Hence the eigenconstants $E_j$ may be calculated using (10).

However, for $\epsilon = Sh_w$, $\lambda_0^2 = 0$ and an expression of the form of (37) for $\gamma_0$ cannot be found in this case. It is possible to obtain series for $\gamma_0$, $Y_0$, and $Y_0'$ in terms of powers of $\epsilon^{1/2}$ and hence calculate $E_0$ from (10), but it was found to be more accurate in this case to write

$$E_0 = E_{00} + \epsilon E_{01} + \epsilon^2 E_{02} + \ldots \quad (39)$$

and to calculate $E_0$ using Equation (9).
From (9)

$$E_0 \int_0^1 (1-\eta^2)M_0^2(\eta)d\eta = \int_0^1 (1-\eta^2)M_0(\eta)d\eta \quad (40)$$

Substituting for $E_0$ from (39) and $M_0(\eta)$ from (24) and comparing like powers of $\epsilon$ gives a system of equations from which $E_{00}$, $E_{01}$, etc. may be calculated. The integrals in this system of equations can be evaluated by using the usual orthogonality relationships together with Equations (26) to (28) with $j = 0$ and $\lambda_0 = 0$.

The above method was also applied for general $E_j$, but the eigenconstants thus obtained were found to be less accurate than those found from (10).

## DISCUSSION

Eigenvalues and eigenfunctions were obtained from Equations (15) and (14) for selected values of wall Sherwood number and were compared with results found by previous workers. The results were found to agree, higher eigenvalues being obtained to more decimal places than in previous papers. The eigenvalues and eigenfunctions were calculated by computer with no difficulty in obtaining higher eigenvalues either by failure of the eigenvalue to converge or by using large amounts of computer time as experienced by previous workers. In fact, once the interval in which the eigenvalue occurs was found, only a small number of iterations was needed to obtain the eigenvalue to six places of decimals. The algorithms for producing the eigenvalues and eigenfunctions are readily transferable for use on an electronic desk calculator; the eigenfunction being particularly easy to evaluate once the eigenvalue is known. It should be noted that in comparing results with those obtained by other workers (for example, Colton et al., 1971), we have used $2h$ instead of $h$ as the channel height and other definitions have been altered accordingly.

It is particularly interesting to note that solutions involving the confluent hypergeometric function can also be found for problems of heat and mass transfer in a circular pipe. The relevant equations are given in Appendix I.

Instead of evaluating Equations (14) and (15) for each value of $Sh_w$ it is possible to obtain solutions for $\gamma$ and $M(\eta)$ by substituting for $Sh_w$ in the relevant power series. For small $Sh_w$ the power series was found to be accurate for $0 \leq Sh_w \leq 0.5$, whereas the series for large $Sh_w$ was found to be accurate for $Sh_w \geq 10$. Indeed the series for large $Sh_w$ gave good approximate solutions for values of wall Sherwood numbers as low as $Sh_w = 1$ and for small $Sh_w$ the series gave good approximate solutions for values of wall Sherwood number up to $Sh_w = 1$. In using the tabulated results, it was found that for the first eigenvalue and eigenfunction, the series for large $Sh_w$ produced the better results for as low as $Sh_w = 0.8$. Thereafter, for values of $Sh_w < 2$, it was found that the series for small $Sh_w$ was the more accurate. It is important to note that the overall and fluid-side Sherwood numbers evaluated from the power series solutions were accurate for most values of $Sh_w$ and even in the least accurate range, $0.5 < Sh_w < 5$, were in error by at most 1%. A sample of the results obtained from using the powers series for wall Sherwood numbers of 1, 2, and 5 are shown in Appendix II. The Sherwood numbers and mixing-cup concentration are evaluated at $x = 0.05$ as this provides the most stringent test in the range over which the results are applicable.

The systems of equations for $M_j(\eta)$ and $Y_j(\eta)$ have also been solved analytically, but there is little to be gained over the numerical solutions at present.

The six sets of eigenvalues and eigenfunctions tabulated are sufficient to give the solution for $x \geqq 0.05$. Clearly, for $x < 0.05$ more eigenfunctions are required and these may be easily obtained from the confluent hypergeometric solution.

## NOTATION

$B$ = coefficient in Equation (19)
$c$ = concentration
$D$ = diffusion coefficient
$E$ = eigenconstant
$F$ = minimization function, Equation (15)
$h$ = ½ channel height
$k$ = mass transfer coefficient; also a constant in Equations (29) and (30)
$M$ = eigenfunction
$M_{k,m}$ = confluent hypergeometric function, Equation (12)
$r$ = dimensionless radial coordinate in appendix
$Sh$ = Sherwood number
$U$ = mean velocity
$x$ = dimensionless axial coordinate = $zD/Uh^2$, dummy variable in (11) and (12)
$Y$ = derivative, with respect to $\gamma$, of eigenfunction
$y$ = transverse coordinate
$z$ = axial coordinate

### Greek Letters

$\alpha$ = terms in power series for $\gamma^2$, Equation (25)
$\beta$ = terms in power series for $\gamma$, Equation (37)
$\gamma$ = eigenvalue
$\epsilon$ = small quantity, $Sh_w$ for $Sh_w < 1$, $1/(2 + Sh_w)$ for $Sh_w \geqq 1$
$\eta$ = dimensionless transverse coordinate = $y/h$
$\lambda$ = eigenvalue in power series for $\gamma$ and $\gamma^2$
$\theta$ = dimensionless concentration $(c - c_0)/(c_i - c_0)$

### Subscripts

$f$ = fluid-side
$i$ = initial
$j$ = index of eigenvalues, etc.
$m$ = mixing-cup
$o$ = overall, outside wall
$w$ = wall, that is, at $\eta = 1$

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## APPENDIX I

In a circular pipe the equation for the dimensionless concentration $\theta$ may be written as

$$(1 - r^2)\frac{\partial \theta}{\partial x} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \theta}{\partial r}\right) \quad (1a)$$

with

$$\theta = 1 \quad x \leqq 0 \quad \text{all } r \quad (2a)$$

$$\partial\theta/\partial r = 0 \quad r = 0 \quad \text{all } x \quad (3a)$$

$$-\partial\theta/\partial r = Sh_w\theta \quad r = 1 \quad \text{all } x \quad (4a)$$

Separation of variables gives

$$\theta = \sum_{j=0}^{\infty} E_j M_j(r) e^{-\gamma_j^2 x} \quad (5a)$$

where

$$M_j''(r) + \frac{1}{r}M_j'(r) + \gamma_j^2(1 - r^2)M_j(r) = 0 \quad (6a)$$

subject to

$$M_j(0) = 1 \quad M_j'(0) = 0 \quad (7a)$$

$$-\frac{\partial M_j}{\partial r} = Sh_w M_j \quad \text{at} \quad r = 1 \quad (8a)$$

Putting $x = \gamma_j r^2$ and $M(x) = v(x)/x^{1/2}$ gives

$$v''(x) + \left\{-\frac{1}{4} + \frac{k}{x} + \frac{\left(\frac{1}{4} - m^2\right)}{x^2}\right\} v(x) = 0 \quad (11a)$$

where $k = \gamma_j/4$ and $m = 0$

Thus

$$v(x) = x^{1/2}e^{-1/2x} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2} - \frac{\gamma_j}{4}\right)_n}{(1)_n\, n!} x^n$$

$$\therefore \quad M_j(r) = e^{-\gamma_j r^2/2} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2} - \frac{\gamma_j}{4}\right)_n \gamma_j^n r^{2n}}{(1)_n\, n!} \quad (13a)$$

$$= e^{-\gamma_j r^2/2}\left\{1 + \sum_{n=1}^{\infty} \frac{(2 - \gamma_j)\ldots(4n - 2 - \gamma_j)\gamma_j^n r^{2n}}{2^{2n}(n!)^2}\right\} \quad (14a)$$

## APPENDIX II.

**Comparison Between Confluent Hypergeometric and Power Series Solutions**

| C.H. solution Eigenvalues | C.H. solution Sherwood numbers and $\theta_m$ at $x = 0.05$ | Power series solution Eigenvalues | Power series solution Sherwood numbers and $\theta_m$ at $x = 0.05$ |
|---|---|---|---|
| $Sh_w = 1$ | | | |
| 1.0 | $Sh_o = 0.729651$ | 1.000843 | $Sh_o = 0.731372$ |
| 4.656137 | | 4.655097 | |
| 8.562020 | $Sh_f = 2.69892$ | 8.561968 | $Sh_f = 2.72262$ |
| 12.515834 | | 12.515815 | |
| 16.487648 | $\theta_m = 0.961772$ | 16.487604 | $\theta_m = 0.951367$ |
| 20.468354 | | 20.468229 | |
| $Sh_w = 2$ | | | |
| 1.220150 | $Sh_o = 1.13696$ | 1.220269 | $Sh_o = 1.141064$ |
| 4.880902 | | 4.875175 | |
| 8.749982 | $Sh_f = 2.63479$ | 8.764895 | $Sh_f = 2.656924$ |
| 12.677288 | | 12.593927 | |
| 16.630025 | $\theta_m = 0.9383585$ | 16.543274 | $\theta_m = 0.937276$ |
| 20.596391 | | 20.603113 | |
| $Sh_w = 5$ | | | |
| 1.445974 | $Sh_o = 1.68056$ | 1.445977 | $Sh_o = 1.68089$ |
| 5.205500 | | 5.205331 | |
| 9.075228 | $Sh_f = 2.53139$ | 9.077524 | $Sh_f = 2.53214$ |
| 12.987426 | | 12.985926 | |
| 16.922820 | $\theta_m = 0.903222$ | 16.873229 | $\theta_m = 0.903221$ |
| 20.872707 | | 20.829852 | |

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# Trajectory Calculation of Particle Deposition in Deep Bed Filtration

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## Part I. Model Formulation

The packed bed model developed by Payatakes, Tien, and Turian (1973a, 1973b) is used as a basis for the study of particle deposition in deep bed filtration. The size of the particulate matters present in the suspension is assumed to be sufficiently large for Brownian motion to be negligible, but small enough for straining to be unimportant. The prediction of the rate of particle deposition is based on the one-step trajectory approach. The collector is represented by a unit bed element of the porous media model and the particle trajectory equation is formulated to include the gravitational force, the hydrodynamic force and torque (including the correction for the presence of the unit cell wall), the London force (including the retardation effect, which is shown to be of primary importance under conditions usually met in deep bed filtration systems), and the electrokinetic force. Sample capture trajectories, including the limiting capture trajectories, are given.

Based on the limiting trajectories and the assumption of uniform particle distribution at the entrance of each unit cell, the number fractions (of suspended particles) impacted on each unit cell are determined and then used to calculate the fraction impacted on the entire unit collector and also the value of the filter coefficient for a clean bed. It is also shown how the capture trajectory calculation can be used to determine the local rate of deposition along the wall of a given unit cell.

### SCOPE

The study reported in this paper represents the beginning phase of a long-range investigation whose objective is the development of a model for the filtration of a liquid suspension through a granular packed bed (or deep bed filtration). The recently proposed P-T-T (Payatakes, Tien, Turian) model for granular porous media is believed to provide a realistic basis for the modeling of the dynamic

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